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Research on a Fractional Exponential Equation

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***Abstract*: Based on a new multiplication of fractional analytic functions, this paper studies a fractional exponential equation. We can find the solution of this fractional exponential equation by using some methods. In fact, the solution is a generalization of the solution of classical exponential equation.**

***Keywords*: New multiplication, Fractional analytic functions, Fractional exponential equation.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. With the development of computer technology, fractional calculus is widely used in various fields of science and engineering, such as physics, mechanics, electrical engineering, viscoelasticity, economics, bioengineering, and control theory [1-10]

In this paper, based on a new multiplication of fractional analytic functions, we study the following α-fractional exponential equation:

$$49^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} + 119^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} = 289^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}, \quad (1)$$

where $0 < \alpha \leq 1$, $\Gamma(\ \ )$ is the gamma function. Using some techniques, the solution of this α-fractional exponential equation can be obtained. On the other hand, our result is a generalization of ordinary exponential equation result.

II. PRELIMINARIES

Definition 2.1 ([11])**:** Assume that x and a_k are real numbers for all k, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic function.

Next, a new multiplication of fractional analytic functions is introduced.

Definition 2.2 ([12])**:** Suppose that $0 < \alpha \leq 1$. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}, \quad (2)$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \quad (3)$$

Then

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$$f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)$$
$$=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}$$
$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)x^{k\alpha}. \tag{4}$$

Equivalently,

$$f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)$$
$$=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{5}$$

Definition 2.3 ([13])**:** If $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n}=f_\alpha(x^\alpha)\otimes\cdots\otimes f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

Definition 2.4 ([14])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}, \tag{6}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{7}$$

We define the compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{8}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{9}$$

Definition 2.5 ([15])**:** Let $0<\alpha\le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{10}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.6 ([16])**:** If $0<\alpha\le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{11}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Definition 2.7 ([17])**:** Let $0<\alpha\le 1$. If $u_\alpha(x^\alpha)$, $w_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}$ is defined by

$$u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}=E_\alpha\left(w_\alpha(x^\alpha)\otimes Ln_\alpha\left(u_\alpha(x^\alpha)\right)\right). \tag{12}$$

Definition 2.8 ([16])**:** Let $0<\alpha\le 1$, and $a_\alpha>0$, $a_\alpha\ne 1$. Then

$$a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}=E_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes Ln_\alpha(a_\alpha)\right)=E_\alpha\left(Ln_\alpha(a_\alpha)\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \tag{13}$$

is called the α-fractional exponential function based on a_α .

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Definition 2.9 ([16])**:** Let $0 < \alpha \le 1$, and $a_\alpha > 0, a_\alpha \ne 1$. Then we define $Log_{a_\alpha}(x^\alpha)$ is the inverse function of $a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}$. In particular, $Log_{e_\alpha}(x^\alpha) = Ln_\alpha(x^\alpha)$.

III. MAIN RESULT

In the following, we find the solution of a fractional exponential equation. At first, we need a lemma.

Lemma 3.1: *Let* $0 < \alpha \le 1$, *and* $p, q > 0$, *then*

$$(p \cdot q)^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} = p^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \otimes q^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}, \tag{14}$$

and

$$\left(\frac{p}{q}\right)^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} = p^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \otimes \left[q^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}\right]^{\otimes -1}. \tag{15}$$

Proof

$$p^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \otimes q^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}$$

$$= E_\alpha\left(Ln_\alpha(p) \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \otimes E_\alpha\left(Ln_\alpha(q) \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= E_\alpha\left([Ln_\alpha(p) + Ln_\alpha(q)] \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= E_\alpha\left([Ln_\alpha(p \cdot q)] \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= (p \cdot q)^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}.$$

On the other hand,

$$p^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \otimes \left[q^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}\right]^{\otimes -1}$$

$$= p^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \otimes q^{\otimes -\frac{1}{\Gamma(\alpha+1)}x^\alpha}$$

$$= E_\alpha\left(Ln_\alpha(p) \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \otimes E_\alpha\left(-Ln_\alpha(q) \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= E_\alpha\left([Ln_\alpha(p) - Ln_\alpha(q)] \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= E_\alpha\left(\left[Ln_\alpha\left(\frac{p}{q}\right)\right] \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$= \left(\frac{p}{q}\right)^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha}.$$

Q.e.d.

Theorem 3.2: *If* $0 < \alpha \le 1$, *then the solution of the* α*-fractional exponential equation*

$$49^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} + 119^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} = 289^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} \tag{16}$$

is

$$x = \left[\Gamma(\alpha+1) \cdot Log_{\frac{17}{7}}\left(\frac{\Gamma(2\alpha+1) + \sqrt{[\Gamma(2\alpha+1)]^2 + 8 \cdot [\Gamma(\alpha+1)]^2 \cdot \Gamma(2\alpha+1)}}{4 \cdot [\Gamma(\alpha+1)]^2}\right)\right]^{\frac{1}{\alpha}}. \tag{17}$$

Proof Since

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$$(7\cdot 7)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} + (7\cdot 17)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} = (17\cdot 17)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\,. \tag{18}$$

It follows that

$$(7\cdot 7)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\otimes\left[(7\cdot 7)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\right]^{\otimes -1} + (7\cdot 17)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\otimes\left[(7\cdot 7)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\right]^{\otimes -1}$$

$$= (17\cdot 17)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\otimes\left[(7\cdot 7)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\right]^{\otimes -1}\,. \tag{19}$$

By Lemma 3.1, we obtain

$$1+\left(\frac{17}{7}\right)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} = \left[\left(\frac{17}{7}\right)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\right]^{\otimes 2}. \tag{20}$$

Let $\frac{1}{\Gamma(\alpha+1)}t^{\alpha} = \left(\frac{17}{7}\right)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}$, where $t > 0$. Then

$$\left[\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right]^{\otimes 2} - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - 1 = 0\,. \tag{21}$$

That is,

$$\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - 1 = 0\,. \tag{22}$$

Therefore,

$$\begin{aligned} t^{\alpha} &= \frac{\frac{1}{\Gamma(\alpha+1)}+\sqrt{\left[\frac{1}{\Gamma(\alpha+1)}\right]^2+\frac{8}{\Gamma(2\alpha+1)}}}{\frac{4}{\Gamma(2\alpha+1)}} \\ &= \frac{\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}+\sqrt{\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\right]^2+8\cdot\Gamma(2\alpha+1)}}{4} \\ &= \frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot\Gamma(\alpha+1)}\,. \end{aligned} \tag{23}$$

Hence,

$$\begin{aligned} \left(\frac{17}{7}\right)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}} &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} \\ &= \frac{1}{\Gamma(\alpha+1)}\cdot\frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot\Gamma(\alpha+1)} \\ &= \frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot[\Gamma(\alpha+1)]^2}\,. \end{aligned} \tag{24}$$

Thus,

$$Log_{\frac{17}{7}}\left(\left(\frac{17}{7}\right)^{\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}}\right) = Log_{\frac{17}{7}}\left(\frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot[\Gamma(\alpha+1)]^2}\right). \tag{25}$$

So,

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha} = Log_{\frac{17}{7}}\left(\frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot[\Gamma(\alpha+1)]^2}\right). \tag{26}$$

Finally, we get

$$x = \left[\Gamma(\alpha+1)\cdot Log_{\frac{17}{7}}\left(\frac{\Gamma(2\alpha+1)+\sqrt{[\Gamma(2\alpha+1)]^2+8\cdot[\Gamma(\alpha+1)]^2\cdot\Gamma(2\alpha+1)}}{4\cdot[\Gamma(\alpha+1)]^2}\right)\right]^{\frac{1}{\alpha}}. \qquad \text{Q.e.d.}$$

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IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we study a fractional exponential equation. In fact, the fractional exponential equation is a generalization of traditional exponential equation. By some techniques, we obtain the solution of this fractional exponential equation. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in engineering mathematics and fractional differential equations.

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